STEPHENS

THE

SIX-CHORD SPIRAL

BY

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PREFACE

The Six-Chord Spiral is an ordinary multiform compound curve of six arcs of equal length, whose degrees of curvature increase in the order of the natural numbers, and so arranged that the seventh arc always exactly coincides with the main circular curve.

As herein outlined it has several valuable features.

1st. It is perfectly flexible and always fits.

2d. No special tables whatever are required for general use. Hence such tables cannot be lost or mislaid. If desired, special tables of the usual form may be quickly computed from Table IV and formulas (1) and (8).

3d. The spiral is adapted to the curve, and not the curve to the spiral of fixed offset or length, as is the case with table spirals.

4th. Odd curves are as readily fitted as even ones, which saves time and trouble in spiraling old track.

5th. Intermediate transit points may be set at any plus, and do not lead to complex deflection calculations.

6th. The method is quickly grasped, memorized,

and applied by transitmen with no previous knowledge of spirals, being based on what they already know; and the mathematical treatment being elementary throughout.

On location it is not even necessary to run in the six-chord, a terminal curve of half the degree of the main curve and giving the same length as the spiral line being substituted.

In this connection note that curves are usually traced a number of times and by different men before the final centering.

7th. It is perfectly interchangeable with the cubic parabola, the two being, within the common limits of spiraling, practically identical.

It should be noted that no spiral changes its degree of curvature directly with the elevation of the outer rail, when the elevation approach has vertical curves at the beginning and end. In this respect all spirals are misfits.

The importance of a proper length of spiral is dwelt upon, and methods are given to insure consistency in this respect with varying conditions of speed and curve.

Comparisons are made between spirals commonly used, which, with the same conditions, define their relations, not only in length and total angle, but also laterally.

The second part deals with methods for shifting old tracks to make room for spirals, pointing out

that this question is entirely independent of the kind of spiral used.

Acknowledgment is due to Professor Talbot for the method of swinging tangents to make room for spirals, and also the method of formulas (27) and (28) for inserting a spiral between the two arcs of a compound curve (see Talbot's "Transition Spiral").

J. R. STEPHENS.

DENVER, COLORADO,
November 5, 1906.

NOTE. —

Natural versed sines are much used in this book.

When not given in the ordinary field tables, they may be found by mentally subtracting the natural cosine of the given angle from .9999 (10), working from left to right, and calling the last decimal used 10.

To find the angle corresponding to a given natural versed sine, subtract the latter from .9999 (10), as above. The remainder will be the natural cosine of the required angle.

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THE SIX-CHORD SPIRAL.

PART I.

LOCATION AND CONSTRUCTION OF SPIRALS.

There are two general forms of spirals in common use.

1st. The *Track Parabola,* in which the deflections from the point of spiral vary as the squares of the distances measured from the same point along the curve.

With the track parabola, any given values of R_M and p, Fig. 1, are fitted exactly.

Further, any intermediate point can be set exactly, and, the instrument being moved up, work continued in a manner similar to that used in laying out circular curves.

This, however, sometimes results in trouble for inexperienced men.

2d. The *Polychord Spiral,* in which the degree of curve increases with each chord, in arithmetical progression.

The polychord spiral with an infinite number of chords is the track parabola.

Reduced to its simplest form, the polychord becomes what might be called a *One-Chord Spiral.*

The latter is a terminal circular curve having a radius 2 R_M (see dotted curve, Fig. 1).

The values of p and R_M being fixed, all poly-chord spirals will fall between the one-chord spiral

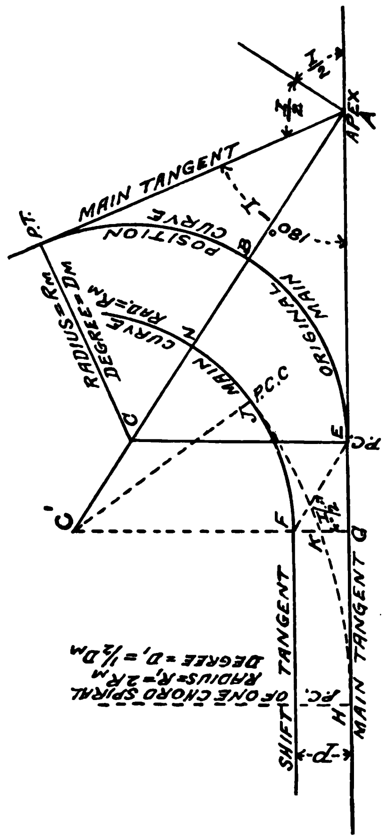

FIG. I.

and the track parabola, and the greater the number of chords, the nearer the approach to the track parabola.

For fixed values of p and R_M, each form of spiral has its own appropriate length, the one-chord being the shortest and the track parabola the longest, all the polychords falling in between; the greater the number of chords the longer the spiral.

In practice, the maximum lateral variation of a six-chord from a parabola will not exceed 0.02 feet. The usual variation is negligible in this class of work. Hence the principal easement curves in use yield alinements which approach each other so closely that their riding qualities are the same.

The total length of track, between common points on the main tangent and main curve, is also the same, no matter what spiral be used, so that, after track is laid to a one-chord, it may be thrown into a track parabola without altering the expansion.

The three principal classes of polychords are:

1st. With deflections constant, while chord length and number of chords vary (such as the Searles form).

2d. With chord length constant, while deflections and number of chords vary.

3d. With number of chords constant, while deflections and chord lengths vary.

Most of these spirals depend for their usefulness on specially prepared tables, which must be con-

sulted in the field, and their efficiency for varying values of p and R_M increases with the number of tables.

Thus, Searles has provided 500 tabulated spirals from which to select the one coming nearest to given values of p and R_M.

The spiral used in the following discussion is of the third type and has invariably six chords.

The *Six-Chord Spiral* is chosen:

1st. On account of its extremely simple relation to the one-chord spiral or terminal arc of half the degree of the main curve (see Fig. 2).

2d. On account of its close approximation to the track parabola, and all polychords commonly used.

It will first be considered as a curve to be offset from the one-chord spiral.

The offsets are small, and may usually be estimated in a manner analogous to the use of the self-reading rod in leveling.

The instrument is to be kept on the one-chord spiral, and all calculations, shifts, etc., are made by the ordinary rules and tables for circular curves.

Notes are kept and plats made precisely as for compound curves.

The one-chord is sufficiently exact for right-of-way descriptions.

Since the one-chord and the six-chord have the same length between common points, no equation

of distance is introduced in passing from one to the other.

To aid the eye in offsetting in the field of view of the instrument, a $2\frac{1}{2}$-inch wrought-iron washer

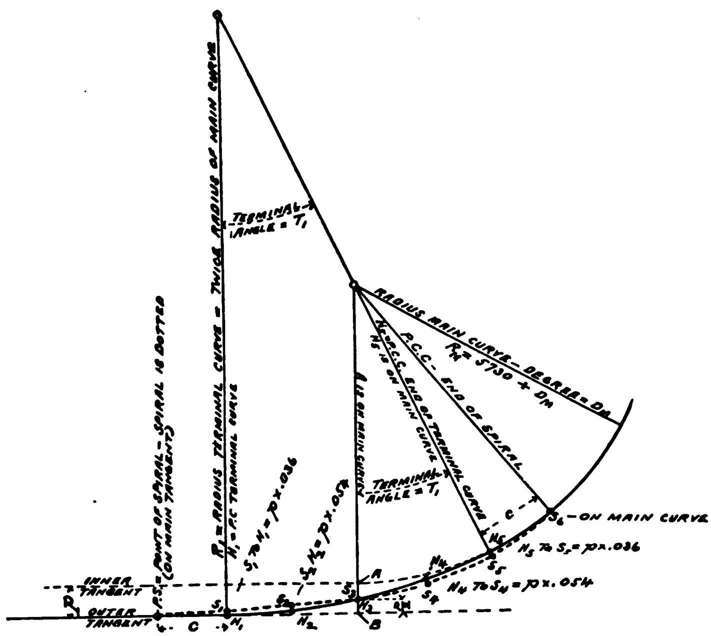

FIG. 2.

may be put on the transit rod. This will give a 0.1 ft. offset on each side of the center of rod, which is usually a sufficient help for setting stakes.

A more exact makeshift may be obtained as follows:

Take a two-foot rule, cut off the two outside hinged legs, thus leaving the pivot joint with a six-inch leg on each side. Screw one of these legs along a face of an ordinary wooden octagon rod.

The other leg will make a folding offset sight. This movable leg should have fastened to its face a strip of sheet-iron, say 6 in. long and 1 in. wide, in which *V*-shaped notches are cut, deep ones for the full tenths from rod center, and shallow for the half tenths.

When the vertical hair cuts the scale at the proper offset, set tack at point of rod.

In case the spiral is so long that a division into six parts gives too great a distance between track centers, it may be divided into twelve equal parts by taking every fifth point in Table I.

This will not constitute the regular twelve-chord spiral, which would be longer and include a greater total angle than the six-chord.

As a guide to section foremen in determining track elevation it is preferable to divide the spiral into some fixed number of equal parts, regardless of the full stationing.

FORMULAS (see Fig. 2).

$$p = R_M (1 - \cos T_1) \tag{1}$$

$$R_M = \frac{p}{1 - \cos T_1} \tag{2}$$

$$\cos T_1 = \frac{R_M - p}{R_M} \tag{3}$$

$$p = .436\, D_1\left(\frac{L_1}{100}\right)^2 \qquad (4)$$

$$L_1 = 151.5\sqrt{\frac{p}{D_1}} = \tfrac{2}{3}\, L_6 \qquad (5)$$

$$L_1 = 2\sqrt{pR_1} = \tfrac{2}{3}\, L_6 \qquad (6)$$

$$p = \frac{L_1^2}{4\, R_1} \qquad (7)$$

The inferiors "M," "1," and "6" indicate respectively "main curve," "one-chord," and "six-chord." L and R are lengths of arc and radius in feet, and D_1 = degree of one-chord.

The Six-Chord Spiral and Terminal Curve Having a Radius Twice that of Main Curve.

This spiral (Fig. 2) has six chords, each one-fourth length of terminal curve, hence spiral is $1\frac{1}{2}$ times length of terminal curve, and the quarter points $H_1 H_2 H_4 H_5$ of the terminal curve, are abreast the one-sixth points $S_1 S_2 S_4 S_5$ of the spiral. S_3 and H_3 coincide. $S_3 A = S_3 B$. One-half the terminal curve is inside the spiral, the other half outside; and the offsets between them, at equal distances from H_3 or S_3, are equal. $H_1 S_1 = H_5 S_5 = .036\ p$, and $H_2 S_2 = H_4 S_4 = .054\ p$. The offset $p = R_M\ (1 - \cos\ T_1) = R_M \times$ versed sine T_1, where R_M = radius of main curve, and T_1 = the terminal angle. (1) $R_M = \dfrac{5730}{D_M}$

To locate the spiral, take the distance for gaining the required elevation $= L_6 = 6C$ (at the nearest multiple of six feet, to avoid fractional chaining).

Here C = chord, and $L_6 = 6C$ = length of spiral.

Then, $$\frac{2C \times D_M}{100} = T_1$$

where $\begin{cases} D_M = \text{degree of main curve.} \\ T_1 = \text{terminal angle in degrees.} \\ C = \text{length of chord in feet.} \end{cases}$

Next calculate p from equation (1) above — run in the terminal curve and offset to spiral. Locate P. S. and S_6, on outer tangent and main curve, one chord-length from H_1 and H_5 respectively.

NOTE. — T_6, the total angle of six-chord $= 1\frac{1}{2}\ T_1$.

Note particularly that the length of six-chord $= L_6$ is 1.5 times the length of the one-chord $= L_1$; also, as an aid to the memory, that the offset .054 = 1.5 times .036.

In practice, taking p at 4 feet, the offsets would be 4 times .054 = 0.216 ft., and 4 times .036 = 0.144 ft.

Example. — Take a 14° curve having a spiral approach of six chords, each 25 ft. long or 150 ft. in all, to connect with a 7° approach, and calculate the offsets to spiral.

$R_M = 5730 \div 14 = 409.3.\quad L_6 = 25 \times 6 = 150.$

T_1 (the terminal angle) $= \frac{1}{3} L_6 \times D = 150 \times 14 \div 3 = 7°$, L_6 being expressed in one hundred-foot units.

The main offset $p = R_M (1 - \cos T_1) = 409.3 \times .00745 = 3.05$ ft.

The offsets

$$H_1S_1 = H_5S_5 = 3.05 \times .036 = 0.11 \text{ ft.}$$
$$H_2S_2 = H_4S_4 = 3.05 \times .054 = 0.16 \text{ ft.}$$
$$H_3S_3 = \text{Zero.}$$

The P. S. and S_6 are set as shown in Fig. 2.

The 7° approach from H_1 to H_5, or the one-chord spiral, will be four 25-ft. chords.

Whenever intermediate offsets are required, as in centering trestle bents, etc., the following table is used:

TABLE I.

Table for Intermediate Offsets to Six-Chord Spiral from Main Tangent and Main Curve with One-Chord Approach.

(To be Measured Inward from the Main Tangent Half of Spiral and Outward from the Main Curve Half).

Tenths of chord length	Coefficients which × p give offsets in feet	Tenths of chord length	Differences for each hundredth of chord length	Tenths of chord length	Coefficients which × p give offsets in feet	Tenths of chord length	Differences for each hundredth of chord length	Tenths of chord length	Coefficients which × p give offsets in feet	Tenths of chord length	Differences for each hundredth of chord length
P.S.	.000	S_6	.0000	S_1	.036	S_5	.0006	S_2	.054	S_4	.0004
1	.000	9	.0001	1	.042	9	.0006	1	.050	9	.0004
2	.001	8	.0002	2	.048	8	.0004	2	.046	8	.0005
3	.003	7	.0003	3	.052	7	.0004	3	.041	7	.0005
4	.006	6	.0003	4	.056	6	.0002	4	.036	6	.0005
5	.009	5	.0004	5	.058	5	.0001	5	.031	5	.0005
6	.013	4	.0005	6	.059	4	.0000	6	.026	4	.0006
7	.018	3	.0005	7	.059	3	.0000	7	.020	3	.0006
8	.023	2	.0006	8	.059	2	.0002	8	.014	2	.0007
9	.029	1	.0007	9	.057	1	.0003	9	.007	1	.0007
S_1	.036	S_5		S_2	.054	S_4		S_3	.000	S_3	

Example. — In the preceding example let the P. S. be at station 7 + 07, chords 25 feet; required the offset at the even station 8.

The curve may be tabulated thus:

$$\left.\begin{aligned} \text{P. S.} &= 7 + 07 \\ S_1 &= 7 + 32 \\ S_2 &= 7 + 57 \\ S_3 &= 7 + 82 \\ S_4 &= 8 + 07 \end{aligned}\right\}$$ Hence $8 = S_3 + \frac{18}{25} = S_3 + 0.72$ toward S_4, which, by interpolation in Table I, equals .042; and

$$.042 \times p \text{ or } 3.05 = .128 \text{ ft.}$$

If the numbering ran in the opposite direction, the offset at 6 + 40 being required, then:

$$\left.\begin{aligned} \text{P. S.} &= 7 + 07 \\ S_1 &= 6 + 82 \\ S_2 &= 6 + 57 \\ S_3 &= 6 + 32 \end{aligned}\right\}$$ Here $6 + 40 = S_3 + \frac{8}{25} = S_3 + 0.32$ toward S_2, which, by Table I, equals

$$.021 \times 3.05 = .064 \text{ ft.}$$

In case a simple curve has been run in connecting the main tangents, as in Fig. 1, no provision being made for spiraling, the circular curve is moved inward, without altering the original radius, along the line BC, for the distance $EF = p \div \cos \frac{1}{2} I$, where p is the principal offset and I the total angle turned between tangents, EF being parallel to BC. Also $EG = p \tan \frac{1}{2} I$.

The distance back to the P. C. from G of the one-chord spiral approach at H is (see Fig. 1)

$$GH = R_M \sin T_1 \tag{8}$$

and

$$EH = R_M \sin T_1 + p \tan \tfrac{1}{2} I \tag{9}$$

$$AH = (R_M + p) \tan \tfrac{1}{2} I + R_M \sin T_1 \tag{10}$$

In order to avoid small equations and to fit the ground from the start, the one-chord spiral should be run in on the first located line that is likely to become final.

Formula for Substituting Spirals Between Two Curves, by Shifting the Position of the Original Tangent to make Room for the Spirals, Leaving Main Curves Undisturbed.

Let A be the angle between the old and new tangents;

L = length of original tangent; p_1 and p_2 = values of principal offsets selected for the two curves respectively;

R_1 and R_2 = radii of the two curves respectively.

Then, when the curves are in opposite directions,

$$A \text{ (in minutes)} = \frac{3440\ (p_1 + p_2)}{L} + \left(\frac{3440\ (p_1 + p_2)}{L}\right)^2 \times .000145\ \frac{(R_1 + R_2)}{L};$$

and when the same curves are in the same direction,

$$A \text{ (in minutes)} = \frac{3440\ (p_2 - p_1)}{L} - \left(\frac{3440\ (p_2 - p_1)}{L}\right)^2 \times .000145\ \frac{(R_1 - R_2)}{L}.$$

Example. — Given alinement as follows:

Zero = P. C. 9° R for 36°.
4 = P. T.
7 = P. C. 6° L for 30°.
12 = P. T.

To insert spirals between the curves:

By Rule 1, page 26, for length of spiral, with speed at $33\frac{1}{3}$ miles per hour, the six-chord for $9^\circ = 33\frac{1}{3} \times 6$ in. elevation = 200 ft.; and six-chord for $6^\circ = 33\frac{1}{3} \times 4$ in. elevation = 133.3 ft.

The lengths of terminal curves are:

133.3 of $4^\circ\ 30'$ for $9^\circ = 6^\circ$, total angle.

88.9 of 3° for $6^\circ = 2^\circ\ 40'$, total angle.

$R_1 + R_2 = 955 + 637 = 1592.$

$637 \times \text{vers } 6^\circ = 3.49 = p_1.$

$955 \times \text{vers } 2^\circ\ 40' = 1.03 = p_2$, and $p_1 + p_2 = 4.52.$

Then, by above formula:

$$A = \frac{3440 \times 4.52}{300} + \left(\frac{3440 \times 4.52}{300}\right)^2 \times .000145 \times \frac{1592}{300}$$

(The original tangent being 300 feet long),

$$A = 51.83' + 2.07' = 53.9' = 54', \text{ approx.}$$

This is 10 feet on 9° curve, and 15 feet on 6°, and the corrected alinement without terminal curves would read:

Zero = P. C. 9° R = $36^\circ\ 54'$.
4 + 10 = P. T.
6 + 85 = P. C. 6° L = $30^\circ\ 54'$.
12 + 00 = P. T.

Then, as one-half of each terminal curve lies either way from the P. T. of 9° and the P. C. of 6°, the new alinement (ignoring the small equation which should be made to fall on the new tangent between the spirals) will be:

Zero	= P.C.	$9°$ R for $30° 54'$, total angle.
3 + 43.3	= P.C.C.	$4° 30'$ R for $6°$, total angle.
4 + 76.7	= P.T.	
6 + 40.6	= P.C.	$3°$ L for $2° 40'$, total angle.
7 + 29.5	= P.C.C.	$6°$ L for $28° 14'$, total angle.
12 + 00	= P.T.	

Compound Curves.

Whenever the degrees of curvature of the two members of a compound curve differ materially, they should be connected by a spiral.

This spiral should be run in on the original location, to save the trouble of subsequent shifts, equations, etc.

The general method before described, of offsets from a one-chord to a six-chord spiral, may be applied equally well in this case.

The one-chord connection averages the degrees of the adjacent main curves.

Thus, a $4°$ compounding into an $8°$ will have a one-chord connection of $\frac{1}{2}(8 + 4) = 6°$.

To make room for this intermediate $6°$, a sufficient offset between the two main curves must be allowed, and the sharper curve must lie inside the lighter one.

The length of the one-chord spiral, the principal offset or gap p, and the intermediate offsets, are determined as follows:

Take the $4°$, $6°$, and $8°$ combination and assume

that the whole curvature is uniformly "bent" outward until the 4° becomes a tangent, the 6° a 2°, and the 8° a 4°.

We then have the conditions of a 4° curve from tangent, and the necessary calculations are made, as before shown, to fit these conditions.

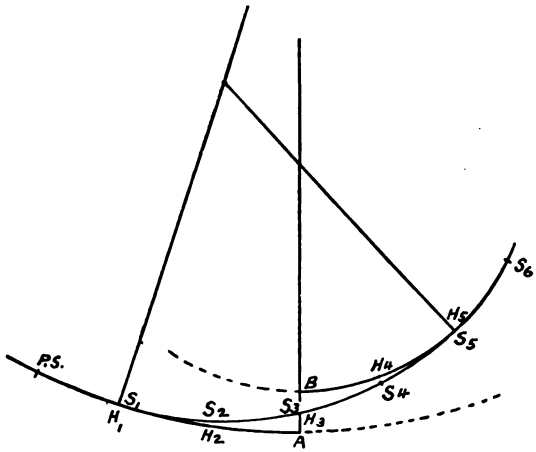

FIG. 3.

P. S. to S_1 = S_5 to S_6 = $\frac{1}{4}$ S_1S_5 = $\frac{1}{4}$ H_1H_5

$AS_3 = AH_3 = BS_3 = BH_3$. $H_1H_3 = H_3H_5$

NOTE.—All "*H*" points are on one-chord spiral; all "*S*" points are on six-chord spiral.

Example.—(See Searles' "R. R. Spiral," page 63, Art. 55.)

Given a compound curve in which $d' = 6°$ and $d'' = 10° \ 40'$, to replace the P. C. C. by a spiral having six chords of 25 ft. each (P. S. to S_6, Fig. 3).

First determine the data for the one-chord, H_1H_5, Fig. 3.

Its degree $d_1 = \frac{1}{2}$ (10° 40′ + 6°) = 8° 20′.

Its length $l_1 = 4 \times 25 = 100$ ft.

Its total angle $t_1 = 8\frac{1}{3} \times 100 = 8° 20′$, of which $d' \times \frac{1}{2}\, l_1 = 6° \times .50 = 3°$ is deducted from the 6°, and $d'' \times \frac{1}{2}\, l_1 = 10° 40' \times .50 = 5° 20'$ is deducted from the 10° 40′.

The total angle of the six-chord spiral will be $8° 20' \times 1.5 = 12° 30'$; of this, $d' \times \frac{3}{4}\, l_1 = 6° \times .75 = 4° 30'$ is deducted from the 6°, and $d'' \times \frac{3}{4}\, l_1 = 10° 40' \times .75 = 8°$ is deducted from the 10° 40′.

Note that in this case the choice of a six-chord spiral in Searles is accidental. The above reasoning would not obtain had any other chord number been chosen.

Now, assuming as before that the 6° curve (the lightest of the three) be bent straight, the 8° 20′ curve becomes a 2° 20′, and the 10° 40′ becomes a 4° 40′.

Hence the conditions are a 2° 20′ one-chord approach from tangent to a 4° 40′ main curve.

The terminal angle for 100 feet of 2° 20′ curve = 2° 20′, and $p_1 = .436 \times 2.33 \times 1 = 1.02$ [see (4), page 9]; or $p_1 = 1228 \times .00083 = 1.02$ [see (1), page 8], which is the value given by Searles, page 65.

Then with the instrument at H_1 or H_5 (each

being two chord lengths or 50 feet from the middle point S_3 or H_3), run in the 8° 20′ one-chord spiral and offset.

$$H_1S_1 = H_5S_5 = 1.02 \times .036 = .037 \text{ ft.}$$
$$H_2S_2 = H_4S_4 = 1.02 \times .054 = .055 \text{ ft.}$$

Intermediate offsets are interpolated from Table I, as before shown.

Since in this particular case the maximum difference between the one-chord and six-chord is but $\frac{3}{4}$ in., the six-chord might well be omitted until it comes to the final adjustment of the track.

Note the direction of the offsets, outward from the one-chord line on sharper curve half, and inward on lighter curve half.

Similarly to the above, the length of the one-chord when p_1 is given may be determined from formulas 3, 5 and 6, page 9, taking 4° 40′ as the main curve.

To Shift the Two Members of a Compound Curve so that Suitable Spirals may be Inserted.

Let *LEF*, Fig. 4, be a compound curve, with *B* and *C* as centers (*b* and *c* being the total angles), which has been run in without provision for spirals.

Required to insert spirals without changing the degree of either branch of the original compound.

The required offsets *p* and *P*, Fig. 4, are to be

taken for spirals having a length suitable for the speed and elevation proposed.

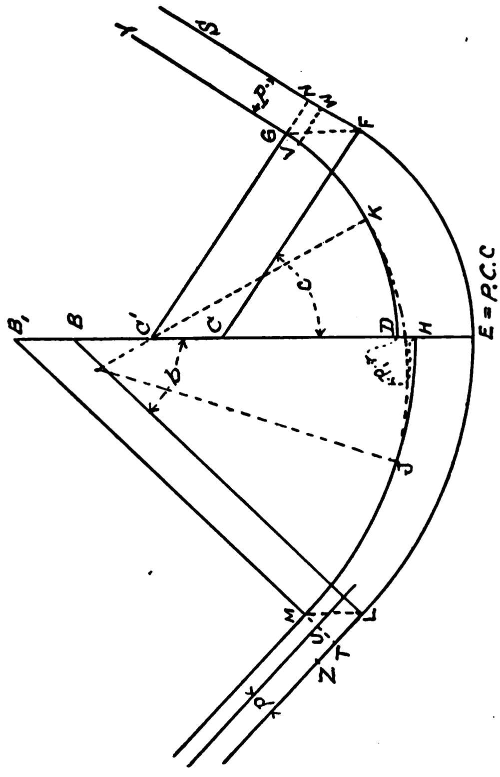

FIG. 4.

Assume that the curve *EF* is slid inward, along the radial line *EB* common to both curves, until *F* falls on *G*, *E* on *D*, and *C* on *C*.′

Then FG, parallel and equal to $ED = \frac{P}{\cos c}$, where $P = GN$ and angle $FGN = c$; also $FN = P \tan c$.

Next determine the proper offset p_1 for a one-chord JK uniting the two members of the compound (see Fig. 3 and following).

$$\text{Then } EH = ED - p_1 = \frac{P}{\cos c} - p_1. \tag{11}$$

Assume that the curve EL is moved inward until E falls on H and L on M, EH being equal and parallel to ML.

Since angle $TML = b$, $MT = ML \cos b$; hence

$$MT = \left(\frac{P}{\cos c} - p_1\right) \cos b. \tag{12}$$

If the curve had been thus run in, the P. T. at M would be a distance, MU, too far out to fit the spiral selected, whose principal offset is p.

To make this fit, the provisional P. C. at G must be pushed ahead along YG produced, for a distance

$$GV = \frac{\left(\frac{P}{\cos c} - p_1\right) \cos b - p}{\sin (b + c)} = WN. \tag{13}$$

If $(b + c)$ exceeds 90°, its sine will be $\sin [180° - (b + c)]$.

$$FW = P \tan c - WN. \tag{14}$$

From W add the distance back to S, making

$WS = CF \times \sin T_1$, where $R_1 = 2\ CF$ (see also equation (8) page 13).

The whole curve, with one-chord spirals, may now be run in, remembering to deduct from the total angle c the terminal angle of its spiral to tangent plus the angle $KC'D$ of the one-chord KJ.

Similarly, the total angle b is reduced by its terminal spiral angle plus the angle $JB'H$.

Angle $KC'D = \frac{1}{2}\, JK \times$ degree of curve EF.
Angle $JB'H = \frac{1}{2}\, JK \times$ degree of curve EL.

In the case of a long compound, minor differences in running may be adjusted by shifting J, the end of the one-chord (see rule, page 56).

This should be done by first running out the full curve JM, and before attempting to put in the final spiral.

In some cases it will be necessary to shift the original P. C. C. before room can be made for end spirals.

In making any or all of these shifts, the nature of the ground should be kept in mind, in order to gain the advantages of a general revision of the line. For this purpose, a large-scale special plat is often of use.

Fig. 5 indicates the process when the curve is to be run in from the lighter end.

Here angle $FGN = b$.

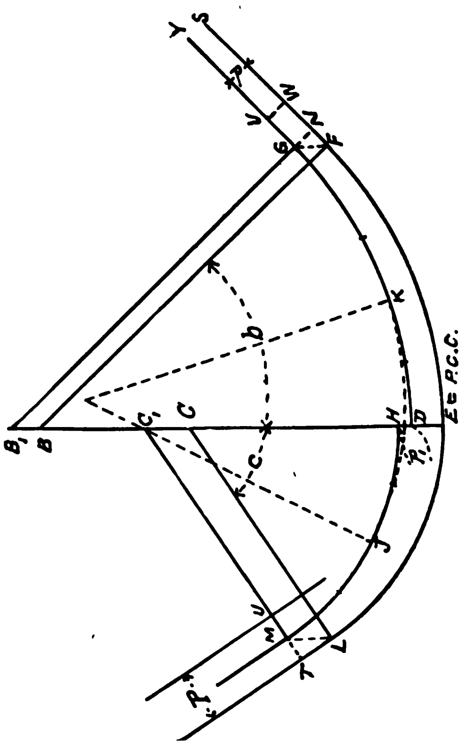

FIG. 5.

Then FG, parallel and equal to ED, $= \dfrac{p}{\cos b}$. (15)

$FN = p \tan b$, $DH = p_1$.

$$EH = ED + p_1 = LM = \frac{p}{\cos b} + p_1. \quad (16)$$

Angle $TML = c$.

Then

$$TM = LM \times \cos c = \left(\frac{p}{\cos b} + p_1\right) \cos c. \qquad (17)$$

$TU = P$, the required offset $= TM + MU$.

Hence the shift required is

$$P - TM = P - \left(\frac{p}{\cos b} + p_1\right) \cos c; \qquad (18)$$

and the necessary pull back is

$$WN = GV = \frac{P - \left(\frac{p}{\cos b} + p_1\right) \cos c}{\sin (b + c)}. \qquad (19)$$

$$FW = FN + WN = p \tan b + WN \qquad (20)$$

The rest of the process is the same as in the preceding case after GV has been obtained.

The Length of Spirals.

There is no definite rule for determining the length of spirals. This depends on both speed and elevation.

The rate on which the given elevation is to be obtained is also important.

Some rules for spiral length are based on a uniform rate of elevation grade, such as 1 in 300, 1 in 400, etc.

The rational rule for varying speeds is that the same amount of super-elevation should be attained in the same time.

This may be called the "*time approach.*"

It follows that curves of the same degree, oper-

ated under different speed conditions, should have spiral lengths proportional to the cubes of the speeds used.

The following tables indicate the relations between spirals, and the data used to determine their lengths. Speeds are in miles per hour. Distances and elevations are in feet.

TABLE II.

Degree Main Curve	Degree One-Chord Spiral	Maximum Safe Speed	Elevation E	Offset p	Length of One-Chord Spiral	Length of Six-Chord Spiral	Rate of Elevation Grade = Speed $\times (6 \div E = 6 \div .5 = 12)$
1	0–30	100.	.5	3.5	400.0	600.0	1 in 1200
2	1–	70.7	.5	3.5	282.8	424.2	1 in 848
3	1–30	57.7	.5	3.5	230.8	346.2	1 in 692
4	2–	50.0	.5	3.5	200.0	300.0	1 in 600
5	2–30	44.7	.5	3.5	178.8	268.2	1 in 536
6	3–	40.8	.5	3.5	163.2	244.8	1 in 490
7	3–30	37.7	.5	3.5	151.0	226.5	1 in 452
8	4–	35.4	.5	3.5	141.6	212.4	1 in 425
9	4–30	33.3	.5	3.5	133.2	199.8	1 in 400
10	5–	31.6	.5	3.5	126.4	189.6	1 in 379

In the above table, the maximum safe speeds are, for convenience, taken as the reciprocals of the square roots of the degrees of main curve $\times$ 100; also,

Length of one-chord spiral = max. speed $\times$ 4;

Length of six-chord spiral = max. speed $\times$ 6.

Note from Table II that for curves operated under the same conditions of safe speed and with

time approaches, the offsets p and the elevation are constant.

The following convenient rules for lengths of spirals are also indicated by the table:

Rule 1. — Length of six-chord equals speed in miles per hour multiplied by elevation in inches.

Here maximum p = 3.5 ft.

Rule 2. — If somewhat longer spirals be desired, then length of one-chord spiral equals speed in miles per hour multiplied by elevation in tenths of feet.

Here maximum p = 5.45 ft.

Other rules, yielding longer or shorter spirals as desired, may be formed on the same plan.

The following table of elevations explains itself. The elevations are in decimals of a foot, and the speeds in miles per hour are given at the heads of the columns.

TABLE III.

Degree of Main Curve	**31.6**	**33.3**	**35.4**	**37.7**	**40.8**	**44.7**	**50.0**	**57.7**	**70.7**	**100.0**
1	.05	.06	.06	.07	.08	.10	.13	.17	.25	.50
2	.10	.11	.13	.14	.17	.20	.25	.34	.50	
3	.15	.17	.19	.21	.25	.30	.38	.50		
4	.20	.22	.25	.29	.33	.40	.50			
5	.25	.28	.31	.36	.42	.50				
6	.30	.33	.38	.43	.50					
7	.35	.39	.44	.50						
8	.40	.44	.50							
9	.45	.50								
10	.50									

Now, finding a 5° curve which is elevated .36 ft. and giving satisfaction as to rail wear, comfort, etc., a glance at Table III shows that it belongs to the 7° maximum series, having a speed of 37.7 miles per hour.

The length of spiral required would be (adopting Rule 1 under Table II), $.36 \times 12 = 4.32$ ins., and $4.32 \times 37.7 = 162.9$ ft. for the length of a six-chord spiral; and $162.9 \times \frac{2}{3} = 108.6$ ft., the corresponding one-chord spiral.

If Rule 2 be adopted, then $10 \times .36 \times 37.7 = 135.72$ = length of one-chord, and $135.72 \times 1.5 = 203.58$ = length of six-chord.

It may sometimes be advisable to use longer easements on certain curves, so that, if the speed limit be increased, the elevation only need be changed, the alinement remaining fixed.

For construction purposes it is necessary to divide the line into speed sections of suitable length, treating each section by itself.

A speed section may sometimes be as short as a single sharp curve, or even the sharp member of a compound curve.

The Length of Spirals Joining Compound Curves.

This should obviously be sufficient to gain the proper difference of elevation between the two curves, or what is the same thing, the length for a spiral from tangent to a curve whose degree is the

difference between the two members of the compound; for example:

A 5° curve compounds with a 3°; required the length of one-chord connection, using Rule 2.

5° − 3° = 2°. Then, assuming speed at 40.8 miles, column 6, Table III, gives elevation for a 2° = .17 ft.

Then 1.7 × 40.8 = 69.4 = length of one-chord spiral. Length of six-chord = 69.4 × 1.5 = 104.1 ft.

To Run in the Six-Chord Spiral by Deflections.

The degrees of curvature of the six arcs of the spiral are:

$$\frac{D}{7}, \frac{2D}{7}, \frac{3D}{7}, \frac{4D}{7}, \frac{5D}{7} \text{ and } \frac{6D}{7}; \frac{7D}{7} = D,$$

being the degree of main curve (see Fig. 2).

The angle of crossing of the six-chord and one-chord at S_3, or H_3, $= \frac{D \times C}{700}$, when both D and the crossing angle are expressed in degrees and decimals, and C equals the length of the single chords in feet.

The total angle of the six-chord $= \frac{D \times L}{200}$.

TABLE IV.

Deflection Coefficients and their Logarithms for Six-Chord Spiral.

These coefficients multiplied by $(C \times D)$, where C equals chord length in feet, and D equals degree of main curve in

degrees, give deflections from tangent at transit in minutes and decimals. Add the logarithms to log $(C \times D)$.

S_7 is on main curve, one chord length beyond S_6, and is given to provide an alternative set-up when S_6 falls on bad ground.

The transit being over any point in the first vertical column, the deflection coefficients are read from this transit point horizontally.

TABLE IV.

Transit over	P.S.	S_1	S_2	S_3	S_4	S_5	S_6	S_7
P.S. coef. . .		.0429	.1071	.2000	.3214	.4714	.6500	.8571
log		8.63202	9.02996	9.30103	9.50708	9.67342	9.81291	9.93305
S_1 coef. . .	.0429		.0857	.1929	.3286	.4929	.6857	.9071
log	8.63202		8.93305	9.28524	9.51663	9.69272	9.83614	9.95768
S_2 coef. . .	.1500	.0857		.1286	.2786	.4571	.6643	.9000
log	9.17609	8.93305		9.10914	9.44494	9.66005	9.82236	9.95424
S_3 coef. . .	.3143	.2357	.1286		.1714	.3643	.5857	.8357
log	9.49733	9.37239	9.10914		9.23408	9.56144	9.76769	9.92206
S_4 coef. . .	.5357	.4429	.3214	.1714		.2143	.4500	.7143
log	9.72893	9.64626	9.50708	9.23408		9.33099	9.65321	9.85387
S_5 coef. . .	.8143	.7071	.5714	.4071	.2143		.2571	.5357
log	9.91078	9.84951	9.75696	9.60975	9.33099		9.41017	9.72893
S_6 coef. . .	1.1500	1.0286	.8786	.7000	.4929	.2571		.3000
log	0.06070	0.01223	9.94378	9.84510	9.69272	9.41017		9.47712
S_7 coef. . .	1.5429	1.4071	1.2429	1.0500	.8286	.5786	.3000	
log	0.18833	0.14834	0.09442	0.02119	9.91833	9.76236	9.47712	
Total angle coef.	P. S. to	0.0857	0.2571	0.5143	0.8571	1.2857	1.8000	2.4000
Total angle log		8.93305	9.41017	9.71120	9.93305	0.10914	0.25527	0.38021

The total angle of the six-chord spiral in minutes $= C \times D \times 1.8$.

The degrees of curvature of the six-chord spiral arcs are $\frac{D}{7}$ to $\frac{6D}{7}$.

p = length of spiral $\times$ sine of deflection angle P. S. to S_3.

See also formula (4), page 9.

Example. — Take a 14° curve having a spiral approach of six chords, each 25 ft. long or 150 ft. in all, to calculate the deflections.

Here $C \times D = 25 \times 14 = 350$. (log = 2.54407).

Then from Table IV, instrument on P. S.,

S_1	S_2	S_3
8.63202	9.02996	9.30103
2.54407	2.54407	2.54407
1.17609	1.57403	1.84510
15′	37.5′	70′
0° 15′	0° 37½′	1° 10′

S_4	S_5	S_6
9.50708	9.67342	9.81291
2.54407	2.54407	2.54407
2.05115	2.21749	2.35698
112.5′	165′	227.5′
1° 52½′	2° 45′	3° 47½′

With instrument at S_6, to turn tangent to the six-chord and main curve at S_6:

Sight on P. S. with vernier set at (see Table IV) $1.15 \times C \times D = 1.15 \times 350 = 402\frac{1}{2}' = 6° \ 42\frac{1}{2}'$, and then turn vernier to zero. Or, sighting on S_3, $0.7 \times 350 = 245' = 4°05'$, which is to be turned off at S_6 to obtain tangent.

These computations may be made by logarithms, as before.

For instrument at S_3 the crossing angle between the spiral and the 7° curve (one-chord

spiral) will be $C \times D \div 700 = 350 \div 700 = 0.5°$ $= 0° 30'$, and from this one may pass from one curve to the other.

The total angle of the six-chord is

$$D \times L \div 200 = 14 \times 150 \div 200 = 10° 30'.$$

To calculate the deflections for a six-chord spiral joining two members of a compound curve (see example under Fig. 3):

First calculate the deflections by Table IV for 150 ft. of six-chord spiral joining a tangent with a $10° 40' - 6° = 4° 40'$ main curve.

Then to each deflection thus found add that of a 6° curve for the length of sight taken.

Thus, from P. S. to S_1 add 45'; from S_3 to S_6 add 2° 15'.

If so desired, necessary tabulations may be prepared in advance, giving once for all the deflections required for the general run of curves in use, precisely as is customary with all table spirals.

The Track Parabola.

Table V may be used in offsetting from the one-chord spiral to the track parabola.

Tables I and V are on the same six-chord base and may be similarly used.

It will be noticed that the differences between the corresponding offsets in Tables I and V are, for any usual value of p, too small to be noteworthy.

In actual service, the parabola has no advantage whatever over the polychord spiral, and a choice between them should be governed by their relative adaptability to field and office use.

The offsets in Table V are to be measured inward from the main tangent half of spiral, and outward from the main curve half.

Note that the offsets at P. S. are insignificant. For $p = 10$ ft. they are 0.01 ft.

TABLE V.

Table of Intermediate Offsets to Track Parabola from Main Tangent and Main Curve with One-Chord Approach.

Tenths of chord length	Coefficients which $\times p$ give offsets in feet.	Tenths of chord length	Differences for each hundredth of chord length	Tenths of chord length	Coefficients which $\times p$ give offsets in feet	Tenths of chord length	Differences for each hundredth of chord length	Tenths of chord length	Coefficients which $\times p$ give offsets in feet	Tenths of chord length	Differences for each hundredth of chord length
P.S.	.001	S_6	.0001	S_1	.038	S_5	.0007	S_2	.055	S_4	.0003
1	.002	9	.0001	1	.045	9	.0006	1	.052	9	.0004
2	.003	8	.0002	2	.051	8	.0004	2	.048	8	.0005
3	.005	7	.0003	3	.055	7	.0003	3	.043	7	.0005
4	.008	6	.0003	4	.058	6	.0002	4	.038	6	.0006
5	.011	5	.0004	5	.060	5	.0001	5	.032	5	.0006
6	.015	4	.0004	6	.061	4	.0000	6	.026	4	.0006
7	.019	3	.0005	7	.061	3	.0001	7	.020	3	.0006
8	.024	2	.0007	8	.060	2	.0002	8	.014	2	.0007
9	.031	1	.0007	9	.058	1	.0003	9	.007	1	.0007
S_1	.038	S_5		S_2	.055	S_4		S_3	.000	S_3	

RELATIVE LENGTHS AND TOTAL ANGLES OF SPIRALS, p AND R_M CONSTANT (see Fig. 1):

Let L_1 = length or total angle of one-chord spiral.
L_6 = length or total angle of six-chord spiral.
L_P = length or total angle of track parabola.

Then $L_6 = 1.5\ L_1 \qquad L_1 = \frac{2}{3}\ L_6 \qquad L_1 = .577\ L_P$
$L_P = 1.733\ L_1 \quad L_P = 1.155\ L_6 \quad L_6 = .866\ L_P$

Example. — Given $R_M = 1432.5 = 4°$ curve,
$p = 4.65$;

The total angle of a one-chord will be [(3), page 8]

$$4° \, 37' = 4.617°$$

The total angle of a six-chord =

$$4.617° \times 1.5 = 6.926° = 6° \, 55\tfrac{1}{2}'$$

The total angle of track parabola =

$$4.617° \times 1.733 = 8° \, 00'$$

Length one-chord = 4.617 ÷ 2 = 230.85 ft.
Length six-chord = 230.85 × 1.5 = 346.28 ft.
Length parabola = 230.85 × 1.733 = 400.00 ft.

These lengths are bisected at S_3, which is the middle point of all spirals.

In the foregoing example, 400 − 230.85 = 169.15 ft. is the difference, $L_P - L_1 = .733\ L_1$.

Hence, 169.15 ÷ 2 = 84.58 ft., is the distance to be laid off along main tangent or main curve from the beginning or ending of the one-chord, in order to obtain the beginning or ending of the track parabola. This may be used in connection

with Table V, when it is desired to lay off the track parabola.

The total angles of the spirals will be divided at the middle point S_8 as follows:

One-chord 4.617°, $\frac{1}{2}$ = 2.31° on tangent half.
One-chord 4.617°, $\frac{1}{2}$ = 2.31° on main curve half.
Six-chord 6.926°, $\frac{2}{7}$ = 1.98° on tangent half.
Six-chord 6.926°, $\frac{5}{7}$ = 4.95° on main curve half.
Track parabola 8°, $\frac{1}{4}$ = 2° on tangent half.
Track parabola 8°, $\frac{3}{4}$ = 6° on main curve half.

In all spiral running it is important to keep a watch on the total angles of the various parts, so that the grand total, from tangent to tangent, will check with the intersection angle of the whole curve.

Demonstration of the Six-Chord Spiral.

In this spiral (Fig. 6), if the total angle of the first arc, P. S. to S_1, be taken as 2, that of the second, S_1S_2, will be 4, the third, S_2S_3, 6, and so on, S_6S_7 being 14. S_6S_7 coincides with the main curve, the end of spiral being at S_6, all chords being of the same length.

Hence the angles which the spiral makes with the outer tangent will be at S_1,S_2, etc., 2, 6, 12, 20, 30, 42, and 56, the angle 42, at S_6, being the total angle of the spiral.

The angle which each chord of the spiral, P. S. S_1, S_1S_2, etc., makes with the outer tangent

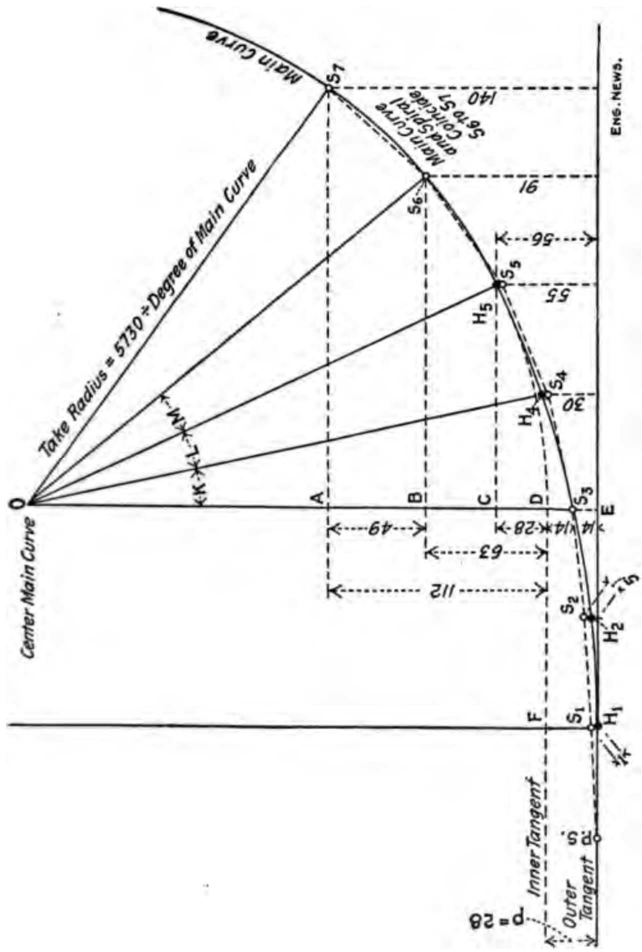
Main Curve
Take Radius = 5730 ÷ Degree of Main Curve
Main Curve and Spiral Coincide S6 to S7
Center Main Curve
Inner Tangent
Outer Tangent
p = 28
ENG. NEWS.
140
91
96
55
30
49
63
112
28
14
14

FIG. 6.

will be the total angle to the end of that chord less the deflection angle of the last arc.

From P. S. to S_1 it equals $2 - 1 = 1$; S_1S_2, $6 - 2 = 4$, and so on, or as the squares of the natural numbers.

Since the sines of small angles are proportional to the angles, the ordinates from S_1, S_2, etc., will be as the sums of these squares, or as 1, 5, 14, 30, 55, 91, and 140, as marked on the figure. $AB = 140 - 91 = 49$.

Since the total angle of the spiral to S_6 is represented by 42, and to S_7 by 56, the angle S_7OS_6 equals $56 - 42 = 14$, both on main curve and spiral. Now, as 14 is one-fourth of 56, continuing the main curve back to D through S_6 and H_5 will make the tangent at D parallel to the outer tangent. The angles K, L, and M each being equal to S_6OS_7, O will be at right angles to DF at D.

Assuming that the versed sines of small angles are proportional to the squares of those angles, we have $AD : BD :: 4^2 : 3^2 = 16 : 9$.

Hence, $AD - BD : BD :: 16 - 9 : 9$. But $AD - BD = 49$, consequently,

$49 : BD :: 7 : 9$. $\therefore BD = 63$, and $AD = 112$.

Take H_5 on the main curve, so that S_6H_5 subtends the angle M and equals S_6S_7; then $AD : CD :: 4^2 : 2^2$, and $CD = 28 = \frac{1}{4} AD$. Also $ED = 140 - 112 = 28$, and $DS_3 = 14 = S_3E$.

Now a circle of twice the radius OS_7, tangent at H_5, will in 4 chord-lengths have a versed sine =

CE or $28 \times 2 = 56$, and be tangent to the outer tangent at H_1.

Taking the ordinates to this circle proportional to the square of the number of chords, it will pass through S_3, and the ordinates to it will be at $H_1 =$ zero, at $H_2 = \frac{56}{16} = 3\frac{1}{2}$, $H_4 = \frac{9}{16} \times 56 = 31\frac{1}{2}$. Hence $H_1S_1 = 1$, $H_2S_2 = 5 - 3\frac{1}{2} = 1\frac{1}{2}$, $H_4S_4 = 31\frac{1}{2} - 30 = 1\frac{1}{2}$, and $H_5S_5 = 56 - 55 = 1$, or, in terms of the main offset, $p = 28$, $H_1S_1 = H_5S_5 = .036\ p$, and $H_2S_2 = H_4S_4 = .054\ p$.

Comparative Tabulations Showing the Relation Between the Six-Chord Spiral and Terminal Curve when Each is Exactly and Independently Calculated.

The following tables give the coördinates of the H points and the S points, by corresponding pairs, on three typical spirals. In each case the spiral and terminal curve are taken to run in a northwesterly direction from a main tangent running due north. For convenience in taking out sines and cosines from table direct, each chord is 100 feet long. Other spirals having the same total angle may be formed by multiplying the tabular quantities by the selected chord length $\div$ 100. In this case the degrees of the main and terminal curves will equal the degrees given in the tables $\div \dfrac{\text{chord length}}{100}$. The bearing of the tangent to curve at each point is also given.

The differences between corresponding pairs of points S_1 and H_1, S_2 and H_2, etc., are taken from each H point as an origin or zero; thus the difference between H_2 and S_2 (in Table VI) of W. .437 and S. .002 means that S_2 lies west and south of H_2, .437 and .002 feet respectively.

TABLE VI.

COÖRDINATES FOR 600 FEET OF SIX-CHORD-SPIRAL APPROACH TO 2° 20′ MAIN CURVE, AND ALSO FOR 400 FEET OF 1° 10′ TERMINAL CURVE JOINING SAME TANGENT AND CURVE.

R_M = 2455.7 ft., p = 8.15 ft., spiral angle = 7°, terminal angle = 4° 40′. Spiral angle of corresponding track parabola = 8° 05′.

$p \times .036 = .293$ ft.; $p \times .054 = .44$ ft.

Point	Bearing of Tangent	Departure	Latitude	Point	Bearing of Tangent	Departure	Latitude
H_1	N	0.000	100.000	H_4	N 3° 30′W	9.160	399.818
S_1	N 20′W	.291	100.000	S_4	N 3° 20′W	8.726	399.851
		W.291	0.000			E.434	N.033
H_2	N 1° 10′W	1.018	199.995	H_5	N 4° 40′W	16.281	499.564
S_2	N 1° W	1.455	199.993	S_5	N 5° 00′W	15.992	499.587
		W.437	S.002			E.289	N.023
H_3	N 2° 20′W	4.072	299.948	H_6	N 7° W	26.445	599.046
S_3	N 2° W	4.073	299.959	S_6	N 7° W	26.445	599.039
		W.001	N.011			0.000	S.007

TABLE VII.

COÖRDINATES FOR 600 FEET OF SIX-CHORD SPIRAL APPROACH TO 4° 40′ MAIN CURVE, AND ALSO FOR 400 FEET OF 2° 20′ TERMINAL CURVE JOINING SAME TANGENT AND CURVE.

R_M = 1228.1 ft., p = 16.26 ft., spiral angle = 14°, terminal angle = 9° 20′. Spiral angle of corresponding track parabola = 16° 10′.

$p \times .036 = .585$ ft.; $p \times .054 = .878$ ft.

Point	Bearing of Tangent	Departure	Latitude	Point	Bearing of Tangent	Departure	Latitude
H_1	N 0°W	0.000	100.000	H_4	N 7°W	18.305	399.274
S_1	N 40′W	.582	99.998	S_4	N 6° 40′W	17.438	399.401
		W.582	S.002			E.867	N.127
H_2	N 2° 20′W	2.036	199.979	H_5	N 9° 20′W	32.510	498.260
S_2	N 2°W	2.909	199.971	S_5	N 10° 00′W	31.931	498.345
		W.873	S.008			E.579	N.085
H_3	N 4° 40′W	8.141	299.792	H_6	N 14° 00′W	52.732	596.194
S_3	N 4°W	8 143	299.834	S_6	N 14° 00′W	52.722	596.160
		W.002	N 042			E.010	S.034

TABLE VIII.

Coördinates for 600 feet of Six-Chord Spiral Approach to 7° Main Curve, and also for 400 Feet of 3° 30′ Terminal Curve Joining Same Tangent and Curve.

$R_M = 819.9$, $p = 24.35$ ft., spiral angle $= 21°$, terminal angle $= 14°$. Spiral angle of corresponding track parabola $= 24° \, 15'$.

$p \times .036 = .877$ ft.; $p \times .054 = 1.315$ ft.

Point	Bearing of Tangent	Departure	Latitude	Point	Bearing of Tangent	Departure	Latitude
H_1	N 0° W	0.000	100.000	H_4	N 10°30′W	27.416	398.369
S_1	N 1° W	.873	99.996	S_4	N 10°W	26.126	398.654
		W.873	S.004			E1.290	N.285
H_2	N 3°30′W	3.054	199.953	H_5	N 14°W	48.634	496.092
S_2	N 3°W	4.363	199.935	S_5	N 15°W	47.770	496.284
		W1.309	S.018			E.864	N.192
H_3	N 7°W	12.204	299.533	H_6	N 21°W	78.705	591.464
S_3	N 6°W	12.209	299.627	S_6	N 21°W	78.672	591.390
		W.005	N.094			E.033	S.074

An inspection of these tables shows:

1st. That in all three cases the six-chord spiral practically passes through H_3. In Table VIII (an extreme case of high values for p and spiral angle)

S_3 is W. .005 and N. .094 of H_3, and the tangent to the curve at S_3 bears N. 6° W. Tracing the six-chord south for .094 of latitude would reduce its departure .094 × tangent 6° (.105) = .010, which would cause the six-chord to pass .010 − .005 = .005 feet due east of H_3.

In Table VII S_3 would fall .001 feet due east of H_3.

2d. That in all three cases, the six-chord spiral (continued) practically passes through H_6, which is on the main curve one chord length beyond H_5.

Thus, in Table VIII, S_6 lies E. .033 and S. .074 feet of H_6, and the tangent to the curve bears N. 21° W. A continuation along this tangent for N. .074 feet would make a westing of .074 × tangent 21° (.38) = .028 feet, and the six-chord would pass .033 − .028 = .005 feet due east of H_6.

It is to be noted that continuing the six-chord .074 north would lengthen it along the 7° curve .074 ÷ cos 21° (.93) = .08 feet, thus increasing the total angle to the point abreast of H_6 by 7° × .6 × .08 = $\frac{1}{3}$ minute, which, in this extreme case, would be the error in total angle.

Note also that the coördinates of S_6 divided one by the other give 78.672 ÷ 591.39 = .13303 = tangent 7° 34$\frac{2}{3}$′. Now the table of deflections for a six-chord previously given shows a deflection from P. S. to S_6 of $C \times D \times 0.65 = 100 \times 7° \times$

$0.65 = 455' = 7° 35'$. Here also is an error of $\frac{1}{3}$ minute.

A tabulation similar to VIII but reversed, *i.e.*, starting from S_6 and running back to the P. S., gives for the quotient of the coördinates of P. S., $138.492 \div 580.303 = .23865 =$ tangent $13° 25\frac{1}{3}'$. By table of deflections this angle is $C \times D \times 1.15 = 100 \times 7 \times 1.15 = 13° 25'$, or again an error of $\frac{1}{3}$ minute.

Similar computations will show that the errors for all intermediate deflections are insignificant.

The same treatment of Tables VI and VII will show no material error whatever, that in Table VII from P. S. to S_6, or S_6 to P. S., being only $\frac{1}{10}$ of one minute.

3d. A comparison of the actual offsets between the two curves at H_1, H_2, H_4, and H_5 is best made by platting the coördinates of the S points with reference to their corresponding H points, on a scale of ten inches to the foot, and drawing the tangents through each pair of points from the bearings given in the tables. By this it will be found that in every case (measuring at right angles to the H line) the coefficients .036 and 0.54 multiplied by p will give the correct distance between the two curves, almost exactly.

From the foregoing the conclusion is drawn that, even for unusually large values of p and the spiral angle, the method of offsets from the ter-

minal curve to the six-chord spiral is practically exact, and that the methods of offsets and deflections are interchangeable, *i.e.*, one method will duplicate the other theoretically much closer than either can be made to duplicate itself on the ground, with the customary appliances and methods.

Comparison of Spirals and Summary.

The railroad spiral provides for a gradual change from the position of car and trucks on a tangent to that assumed by them on a curve.

This change is effected by an intermediate curve having an average curvature usually one-half that of the main curve.

Figure 1 shows the general problem. Here FJN is the main curve with center at C', and HGE the main tangent. The main curve has been moved inward a distance BN from its original position. This shift is necessary to allow room for the insertion of the lighter intermediate curve. The new main curve merges into the shift tangent (which is parallel to the main tangent) at F.

The simplest form of spiral is that shown by the dotted curve HKJ, which has twice the radius or half the degree of the main curve. This is called the terminal curve or one-chord spiral. The point K, which practically bisects the principal offset, $FG = p$, is the middle point of the length of the spiral.

H is the P. C. and J the P. C. C. of the one-chord.

If two curves be used in passing from tangent to curve, the degree of the first will be from tangent to $K = \frac{1}{3}$ degree of main curve, and of the second from K to main curve $= \frac{2}{3}$ of the same. As before, K is the half-way point of the spiral. This constitutes a two-chord spiral.

Calling the total length of the one-chord unity, that of the two-chord will be 1.225. Hence the latter starts from the tangent to the left of H, and passing through K merges into the main curve between J and N. It thus lies inside the one-chord from tangent to K, and outside from K to main curve, the two spirals crossing each other at K.

This condition is indicated by the dotted spiral in Fig. 2, where H_1, H_3, and H_5 are points on the one-chord and respectively correspond with H, K, and J of Fig. 1.

Spirals having any number of chords (N) are so taken that the degree of curve of the first arc = degree of main curve $\div (N + 1)$, that of the second twice, of the third thrice that of the first arc, and so on, the $(N + 1)$ arc coinciding with the main curve.

The lengths of spirals for fixed values of p and main curve increase with the number of chords or arcs used; that is, they start further back on the tangent, and, passing through the common point

K (where they are bisected), reach further around the main curve toward N, Fig. 1, before merging into it.

The greater the number of arcs in a spiral, the greater the lateral deviation from the one-chord on the inside of KH and the outside of KJ.

The limit is reached when the number of arcs becomes infinite. The spiral then increases uniformly in curvature from start to finish and without pause. This constitutes the usual track parabola whose length is 1.733 that of the one-chord.

The curvature of all spirals increases at the same rate from K toward the main curve as it decreases from K toward the main tangent.

Hence, with degree of main curve and p fixed, the total angle of a spiral is proportional to its length.

The total angles and lengths of various spirals are given in the following table, those of the one-chord being unity:

1-chord = 1.000	10-chord = 1.581
2-chord = 1.225	11-chord = 1.593
3-chord = 1.342	12-chord = 1.604
4-chord = 1.414	13-chord = 1.613
5-chord = 1.464	14-chord = 1.621
6-chord = 1.500	15-chord = 1.628
7-chord = 1.528	16-chord = 1.634
8-chord = 1.549	17-chord = 1.640
9-chord = 1.567	parabola = 1.733

Example. — Take $R_M = 286.5$ ft. (20° curve) and $p = 4.35$ ft., the total angle of the one-chord being 10°.

These conditions will be fitted by:

100 feet of one-chord, total angle 10°.

150 feet of six-chord, total angle 15°.

156.7 feet of nine-chord, total angle 15° 40′.

173.3 feet of parabola, total angle 17° 20′.

Each chord of the six-chord will be $150 \div 6 =$ 25 ft., and of the nine-chord, $156.7 \div 9 = 17.4$ ft.

The degrees of curve of the six arcs of the six-chord will be $\frac{20}{7}$, $\frac{40}{7}$, $\frac{60}{7}$, $\frac{80}{7}$, $\frac{100}{7}$, and $\frac{120}{7}$. The seventh or $(N + 1)$ arc is $\frac{140}{7}$, which is the 20° main curve. In this example, the difference (173.3 − 150) divided by 2 (= 11.65 ft.) is the amount the parabola overlaps the six-chord at each end.

The lateral variation of any of these spirals from the one-chord or from each other is the same at equal distances from K measured along the spiral, but these offsets are to be made inward from the one-chord on the main tangent side of K, and outward on the main curve side.

From this it follows that the total length of track between common points on the main tangent and main curve is the same for fixed values of R_M and p, no matter what spiral be used, so that after track has been laid to a one-chord it may be shifted to a track parabola or any intermediate spiral without altering the expansion.

For any one form of spiral with a fixed value of R_M, the principal offset p varies as the square of the length of the spiral; that is, doubling the length of spiral increases p four times.

If the distances along the one-chord from the middle point K or from either end be expressed in fractions of the length of the one-chord, then the offsets from the one-chord *to any fixed form of spiral* at any given point will equal $p \times$ constant coefficient for that point, regardless of the degree of main curve or length of spiral. Thus, in Fig. 2, the offsets S_2H_2 or H_4S_4, which are at the quarter points of the one-chord, will always for a six-chord spiral equal $p \times .054$. From the same quarter points of the one-chord to the parabola the offsets are always $p \times .055$.

The complete coefficients for the six-chord and track parabola are given in Tables I and V; see also Fig. 2.

Since the length of the six-chord is always 1.5 times that of the one-chord, the quarter points of the one-chord lie abreast of the sixth points of the six-chord. Both Tables I and V give the offsets at the various points along the six-chord, from its beginning at P. S. through S_1, S_2, etc., to its end at S_6. This is solely for convenience in setting off and in making comparisons.

These coefficients are the offsets in feet when $p = 1$ ft. For any other value of p, multiply by p.

Thus, when $p = 10$ ft. (an unusually large value), Table I shows that the maximum distance from the six-chord to the one-chord is $.059 \times 10 = .59$ ft. at 1.7 chord lengths from either the beginning or end of the six-chord.

Table V shows that the maximum distance of the parabola from the one-chord is at 1.65 chord lengths from the beginning or end of the six-chord, and equals (with $p = 10$ ft.) $0.061 \times 10 = .61$ ft.

Comparing I and V shows that the greatest divergence of the parabola from the six-chord occurs at 1.2 chord lengths from the beginning or ending of the six-chord and equals $(.051 - .048) \times 10 = 0.03$ ft.

Table V also shows that the offset from main tangent and main curve at the beginning and ending of the six-chord (at P. S. and S_6) equals $p \times 0.001$, which, when $p = 10$ ft., becomes .01 ft.

Hence, for easement purposes, the excess of length of the parabola over the six-chord is negligible.

PART II.

SPIRALING OLD TRACK.

Spiraling old track consists mainly in compounding to make room for the spirals.

The methods used for the shifts are entirely independent of the form of spiral, for, with fixed values of p and R_M, any spiral from the one-chord to the track parabola may be inserted, differing from each other, of course, in length and total angle, according to Table IX, but all giving practically the same length of line between common points.

In making room for a six-chord spiral, the obvious method is to first provide for a one-chord, remembering that the one-chord radius must be double that of the *revised curve* into which it compounds, and not double that of the existing curve, unless the latter be unchanged.

With this condition imposed, any of the formulas for three-center compound curves may be used direct.

Space-shifts for inserting spirals are usually made according to one or the other of the following assumptions:

1st. To leave as much of the original line undisturbed as circumstances permit.

2d. To preserve the original length of line, thus avoiding numerous equations of distance.

In either case the tangents are usually undisturbed, the necessary changes being confined to the curves.

When working on the first assumption, the following compound-curve formulas are most useful (see following example for application):

$$R_N = R_O - \frac{\text{vers } I}{p} \qquad (21)$$

$$p = (R_O - R_N) \text{ vers } I. \qquad (22)$$

$$\text{Vers } I = \frac{p}{R_O - R_N}. \qquad (23)$$

where R_N = radius of new main curve,
R_O = radius of original main curve,
p = principal offset.

I equals the angle cut out of the R_O-curve and replaced by the R_N-curve.

The degree of the R_N-curve is usually taken from one-tenth to one-fifth greater than the degree of R_O.

The one-chord terminal angle T_1 is determined from vers $T_1 = \frac{p}{R_N}$, and either the one-chord or six-chord run in.

The P. C. of the one-chord, $2R_N$, will be back along the main tangent a distance from the original

$$\text{P. C.} = R_N \sin T_1 - p \cot \tfrac{1}{2} I \qquad (24).$$

Example. — To replace one end of a 4° curve with enough 4° 30′ to give an offset $p = 6.62$ ft.

$$\text{Here vers } I = \frac{p}{R_O - R_N} = \frac{6.62}{159} = .0416.$$

Hence, $I = 16° 35'$; and
$16° 35'$ of $4° = 414.6$ ft., also,
$16° 35'$ of $4° 30' = 368.5$ ft.

Hence, the last 414.6 feet of the 4° is to be replaced by 368.5 feet of 4° 30′ curve.

$$\text{Again, vers } T_1 = \frac{p}{R_N} = \frac{6.62}{1273.6} = .0052$$

$$T_1 = 5° 50.6'$$

$$5° 50.6' \text{ of } 4° 30' = \frac{5.844}{4.5} = 129.87 \text{ ft.,}$$

which in its turn is replaced by

$$129.87 \times 2 = 259.74 \text{ ft. of } 2° 15'$$

one-chord approach.

From the preceding formula this 2° 15′ one-chord will begin on main tangent back from original P. C. a distance =

$$(1273.6 \times .1018) - (6.62 \times 6.862) = 84.2 \text{ ft.}$$

It is clear that in the preceding formula I may be made as large as half the intersection angle of the original curve.

If it be desired to throw the middle of the original simple curve out along a radial offset for a distance

h, then, I_2 being half the intersection angle of the original curve,

$$R_N = R_O + h - \frac{p + h}{\text{vers } I_2} \tag{25}$$

Example:—

Take $\quad I = 60°$, hence $I_2 = 30°$

$R_O = 955.4\ (6°)$.

$p = 4.4$ ft. $\quad h = 0.5$ ft.

Then $\quad R_N = 955.4 + 0.5 - \dfrac{4.4 + 0.5}{.134} = 919.3.$

Hence $\quad R_N = 6°\ 14'$ curve.

Also, $\text{vers } T_1 = \dfrac{p}{R_N} = \dfrac{4.4}{919.3} = .00479.$

$T_1 = 5°\ 36.6'$.

Hence 5° 36.6′ of 6° 14′ curve are to be replaced by 5° 36.6′ of 3° 07′ one-chord approach.

The P. C. of this 3° 07′ one-chord will be back along the main tangent a distance =

$$R_N \sin T_1 - (p + h) \cot \tfrac{1}{2} I_2 \tag{26}$$

from the original P. C.,

or $919.3 \times \sin 5°\ 36.6' - (4.4 + 0.5) \cot .15° =$ 71.60 ft.

It is usually best to run such curves as the above from both ends, making the junction at the middle of the curve or on the radial line through the vertex.

If, in recentering old track, the best-fitting curve should merge into a tangent parallel to, and either inside (i) or outside (o) of the existing

tangent (which is to be maintained), then the amount o must be added to, and the amount i subtracted from, p in formulas (23) to (26).

Example. — To replace part of a 4° curve that merges into a parallel tangent 2 ft. outside the existing tangent, by enough 4° 30′ to make p, with relation to the existing tangent, = 6.62 ft.

As the 2-ft. offset o is outside, equation (23) becomes:

$$\text{vers } I = \frac{6.62 + 2}{R_O - R_N} = \frac{8.62}{159} = .0542$$

$$I = 18° \, 57'$$

18° 57′ of 4° = 473.75 ft. to be replaced by
18° 57′ of 4° 30′ = 421.11 ft.

$$\text{Again, vers } T_1 = \frac{p}{R_N} = \frac{6.62}{1273.6} = .0052$$

$$T_1 = 5° \, 50.6' \text{ of } 4° \, 30' = 129.87 \text{ ft.}$$

to be replaced by 259.74 ft. of 2° 15′ one-chord.

The P. C. of this one-chord will be back on main tangent from the original P. C. 4° a distance of

$$R_N \sin T_1 - (p + o) \cot \tfrac{1}{2} I =$$
$$(1273.6 \times .1018) - 5.992\,(6.62 + 2) = 78 \text{ ft.}$$

If the tangent falls inside of the existing tangent an amount i (less than p) of 2 ft., then

$$\text{vers } I = \frac{6.62 - 2}{R_O - R_N} = \frac{4.62}{159} = .02906.$$

Hence $I = 13° \, 51'$, and
$T_1 = 5° \, 50.6'$ as before.

The distance of the new P. C. 2° 15′ one-chord back from the original P. C. 4° will be

$$(1273.6 \times .1018) - 8.233\,(6.62 - 2) = 91.61 \text{ ft.}$$

If i be made larger, I will become smaller and the 4° 30′ curve will soon be too short to serve as the base for a 2° 15′ one-chord with the given value of p. A lighter curve must then be taken, say 4° 20′, 4° 10′, etc., until, when i becomes equal to p, the 4° curve is connected directly with the tangent by means of the proper length of 2° one-chord.

When i exceeds p, a curve lighter than 4° must be taken. In all cases the total angle I of the terminal branch must be at least $1\frac{1}{2}$ times T_1 in order to make room for the six-chord, and at least 1.733 T_1 for the track parabola.

In addition to the formulas above given, the following rule for shifting the P. C. C. of the last arc of any compound curve (without changing the degrees of curve) in order to offset the last tangent parallel to itself, is of constant use.

Rule (Modified from Shunk's "Field Book," page 101): Divide the required offset by the difference of the radii, and call the quotient Q; call the nat. cosine of the total angle of the located last arc C. Then either $Q + C$ or $Q - C$ will be the nat. cosine of the new last arc, and the difference between the angle whose cos $= C$ and the angle whose cos $= Q \pm C$ gives the required angular shift of the P. C. C.

This angular shift is reduced to feet according to the degree of curve of the *next-to-the-last branch,* and on which it must be used.

It is evident that

1st. To offset the last tangent *out* requires *more* of a *lighter* or *less* of a *sharper* final arc.

2d. To offset the last tangent *in* requires *more* of a *sharper* or *less* of a *lighter* final arc.

3d. *Less* final arc requires *more cosine,* hence use $Q + C$.

4th. *More* final arc requires *less cosine,* hence use $Q - C$. If C be greater than Q, use $C - Q$.

Example. — A 3° compounds into a 5°, which latter has a total angle of 30° 22′. It is desired to throw the final tangent *inward* 34 ft.

Here $R - r = 1{,}910 - 1{,}146 = 764$, and

$$\frac{34}{764} = .0445 = Q.$$

$$\text{nat. cos } 30^\circ\ 22' = \underline{.8628} = C.$$
$$\text{nat. cos } 35^\circ\ 05' = .8183 = C - Q.$$

In this case the tangent is to be thrown *in,* hence *more* of the sharper last arc (5°) is required. Therefore use $C - Q$ = nat. cos 35° 05′.

$$35^\circ\ 05' - 30^\circ\ 22' = 4^\circ\ 43'$$

Since more sharper last arc is required, the P. C. C. must be moved back along the 3° curve 4° 43′ = 157.22 ft.

To throw the tangent out 34 ft., proceed as follows:

$.0445 + .8628 = .9073 = \text{nat. } \cos 24° \ 52'.$
$30° \ 22' - 24° \ 52' = 5° \ 30'.$

Here the P. C. C. must be *advanced* along the 3° produced $5.5° \div 3° = 183.33$ ft.

This rule may be used to shift the P. C. C. of a one-chord spiral. In this case the difference of the radii $= 2R_M - R_M =$ the degree of the main curve, and the final arc of one-chord spiral is always the lighter. Hence, to offset the last tangent *out* requires more one-chord, and for this use $Q - C$. To offset the last tangent *in* requires less one-chord, and for this use $Q + C$.

Since, in this case, the original value of p is always known, it is preferable to add to or subtract from p (as the case may be) the required offset, thus forming a new p which is then used to determine the new T_1 by formula (1).

Compound Curves.

Space may be made at the P. C. C. for a spiral between the two members of a compound curve by employing one of the following methods:

(1) By replacing part of the sharper curve with a still sharper one.

(2) By replacing part of the lighter curve with a still lighter one.

(3) By a combination of (1) and (2), preferably by adding as many minutes to the degree of the sharper curve as are subtracted from the degree of the lighter one.

By the third method, the center of the spiral practically falls on the original P. C. C., and the length of line is unchanged.

First Method.—When sharpening the sharper curve (of degree = D_S) to D_N for a length L_N, the original lighter curve D_L must be produced for a distance L_L, so that tangents at the end of L_N of D_S and the end of L_L of D_L are parallel to each other and p feet apart.

(Inferiors: S = Sharper; L = Lighter; N = New).

$$\text{Then } L_L^2 = 1.15 \frac{(D_N - D_S)\,p}{(D_S - D_L)\,(D_N - D_L)} \tag{27}$$

$$\text{and } L_N^2 = 1.15 \frac{(D_S - D_L)\,p}{(D_N - D_S)\,(D_N - D_L)} \tag{28}$$

Example. — A 2° (D_L) and 8° (D_S) compound, and it is required to insert a spiral, p being taken at 3 ft., the 8° to be changed to an 8° 30′ (D_N); here

$$L_L^2 = 1.15 \frac{(8.5 - 8)\,3}{(8 - 2)\,(8.5 - 2)} = .04423$$

$$L_L = .2103 \text{ Stations} = 21.03 \text{ ft.}$$

$$L_N^2 = 1.15 \frac{(8 - 2)\,3}{(8.5 - 8)\,(8.5 - 2)} = 6.369$$

$$L_N = 2.5237 \text{ Stations} = 252.37 \text{ ft.}$$

Hence, the beginning of the new 8° 30′ (D_N) curve will be back along the 8° curve a distance of 252.37 + 21.03 = 273.4 ft. from the original P. C. C., and the resulting condition is 8° 30′ and 2° main curves parallel to each other at a point

(S_3) 21.03 ft. from the original P. C. C. along the original 2° produced, and distant apart 3 ft.; required to connect them with a

$$\frac{8^\circ\,30' + 2^\circ}{2} = 5^\circ\,15' \text{ one-chord spiral.}$$

This 5° 15′ curve starts from the 2° and ends on the 8° 30′ (or vice versa) at a distance from S_3 (Fig. 3) of

$$\tfrac{1}{2}\,l_1 = \sqrt{\frac{p}{.87 \times d}} \tag{29}$$

where d = the difference between the degrees of the two final curves (8° 30′ − 2°), and l_1 = length of one-chord in stations of 100 ft.

$$\text{Hence, } \tfrac{1}{2}\,l_1 = \sqrt{\frac{3}{.87 \times 6.5}} = .7284 \text{ Stations.}$$

The total length of the 5° 15′ = 72.84 × 2 = 145.7 ft., and its total angle = 7° 39′.

(See also example under Fig. 3.)

Second Method. — When lightening the lighter curve D_L for a length L_N, the original sharper curve D_S must be produced a distance L_S so that tangents at the end of L_N of D_L and the end of L_S of D_S are parallel to each other and p feet apart.

$$\text{Then } L_S^2 = 1.15\,\frac{(D_L - D_N)\,p}{(D_S - D_L)\,(D_S^S - D_N)} \tag{30}$$

$$L_N^2 = 1.15\,\frac{(D_S - D_L)\,p}{(D_L - D_N)\,(D_S - D_N)}. \tag{31}$$

Example. — A 2° (D_L) and an 8° (D_S) com-

pound, and it is required to insert a spiral, p being taken at 3 ft. and the 2° to be changed to a 1° 30′ (D_N).

Here $L_S^2 = 1.15\dfrac{(2-1.5)\,3}{(8-2)\,(8-1.5)} = .04423.$

$L_S = 21.03$ ft.

and $L_N^2 = 1.15\dfrac{(8-2)\,3}{(2-1.5)\,(8-1.5)} = 6.369.$

$L_N = 252.37$ ft.

Hence the beginning of the new 1° 30′ curve will be back along the 2° curve a distance of 252.37 + 21.03 = 273.4 ft. from the original P. C. C., and the resulting condition is 1° 30′ and 8° main curves parallel to each other at a point (S_3) 21.03 ft. from the original P. C. C. along the original 8° produced, and distant apart 3 ft.

Required to connect them with a $\dfrac{8° + 1° \, 30'}{2}$ = 4° 45′ one-chord spiral.

This 4° 45′ curve starts from the 8° and ends on the 1° 30′ (or vice versa) at a distance from S_3 (Fig. 3) =

$$\tfrac{1}{2}\, l_1 = \sqrt{\frac{p}{.87 \times d}},$$

where d is the difference between the degrees of the two final curves (8° − 1° 30′), and l_1 = length of one-chord.

Hence $\tfrac{1}{2}\, l_1 = \sqrt{\dfrac{3}{.87 \times 6.5}} = .7284$ Stations, or 72.84 ft., as before.

The total length of the 4° 45′ one-chord = 72.84 × 2 = 145.7, and its total angle = 6° 55′.

(See also Example under Fig. 3.)

Third Method. — When both the sharper curve is sharpened and the lighter curve lightened by equal amounts, the method is as follows:

Example. — A 2° compounds with a 10°. It is desired to replace 150 ft. of the 2° by a 1° 30′, and 150 ft. of the 10° by a 10° 30′, the increase and decrease being each 30′. Here the line will be thrown both in and out at the P. C. C. for a distance of

$$\tfrac{1}{2}\, p = .87\, K L_N^2, \tag{30}$$

where K is the increase or decrease expressed in degrees and decimals, and L_N the length of change of each curve in stations of 100 feet. In this case

$$\tfrac{1}{2}\, p = .87 \times .5 \times 2.25 = .98 \text{ ft., hence}$$
$$p = .98 \times 2 = 1.96 \text{ ft.}$$

The resulting condition is 1° 30′ and 10° 30′ curves parallel to each other at the original P. C. C. and 1.96 ft. apart. Required to connect them with a $\dfrac{10^\circ\, 30' + 1^\circ\, 30'}{2} = 6^\circ$ one-chord spiral.

As before, this 6° curve starts from the 1° 30′ and ends on the 10° 30′ (or vice versa), at a distance from S_3 (Fig. 3) of

$$\tfrac{1}{2}\, l_1 = \sqrt{\frac{p}{.87 \times d}},$$

where d is the difference between the degrees of the two final curves and l_1 = length of one-chord, or

$$\tfrac{1}{2}\, l_1 = \sqrt{\frac{1.96}{.87 \times 9}} = \sqrt{.25} = .5 \text{ stations.}$$

Hence the 6° will have a total length of $.5 \times 2 =$ 100 ft., or 50 ft. each way from the original P. C. C.

When the offset p is given and also the increase and decrease of the degree of curve, proceed as follows:

Example. — A 2° compounds with a 10°, p is to be taken at 1.96 ft., and 1° 30′ and 10° 30′ curves used.

Here

$$L_N{}^2 = \frac{\tfrac{1}{2}\, p}{.87\, K}, \text{ or } L_N = .758\sqrt{\frac{p}{K}}, \qquad (31)$$

where L_N = length of 10° 30′ or 1° 30′ (to be used measured from original P. C. C.), p = principal offset, and K = increase or decrease of degree of curve expressed in degrees and decimals.

In this case $L_N = .758\sqrt{\frac{1.96}{.5}} = 1.5$ Stations = 150 ft., and l_1 is found as above.

These methods for spiraling compound curves, though approximate, give excellent results in practice.

It is to be remembered that in such cases the spiral notes are not used to replace original records; hence there is no real need of absolute accuracy.

SPACE-SHIFTS PRESERVING THE ORIGINAL LENGTH OF LINE.

As previously indicated, when p and R_M are fixed, the one-chord, six-chord and track parabola all give the same length of line between common points on main tangent and main curve.

Hence, for convenience and simplicity, the one-chord will be considered in the following computations:

Given two tangents intersecting at a fixed angle and joined by simple curves of various radii.

Call the distance from P. C. to P. T. along the tangents, via the vertex, the *tangent route*, and the distance P. C. to P. T., via the curve, the *curve route.*

Then the difference between the tangent and curve routes varies in direct proportion with the radius used.

Further, any two curve routes are of equal length when they are equally less than the tangent route common to both.

On these principles the following solutions are based:

Example. — Given a 4° curve for 70° 30′, to substitute a curve with spirals, retaining the same length of line.

Assume a terminal angle (T_1) of 3° 24′.

First, compute the elements of 3° 24′ of 2° one-

chord on each end of ($70° 30' - 6° 48' =$) $63° 42'$ of 4° main curve.

By formula (1) $p = \text{vers}\, T_1 \times R_M = .00176 \times 1432.7 = 2.52$ ft.

By formula (10) the distance from the apex to P. C. of one-chord is $(R_M + p) \tan \frac{1}{2} I + R_M \sin T_1$ (see Fig. 1).

$R_M + p$	$= 1435.22$	log	$= 3.156918$
$\frac{1}{2} I$	$= 35° 15'$	log tan	$= 9.849254$
GA	$= 1014.31$	log	$= 3.006172$
R_M	$= 1432.7$	log	$= 3.156151$
T_1	$= 3° 24'$	log sin	$= 8.773101$
GH	$= 84.97$	log	$= 1.929252$

$$AH = GA + GH = 1099.28$$

Hence tangent route =

$$2\,(1014.31 + 84.97) = 2198.56$$

By the curve route there is

6° 48′ of 2° = 340 ft., and	
63° 42′ of 4° = 1592.5 ft.: total	= 1932.50
Tangent route less curve route	= 266.06

Now, to preserve the original length of line, this difference must be reduced by shortening the radii until it equals the original difference between the tangent and curve routes. The latter is calculated thus:

R_M	$= 1432.7$	log	$= 3.156151$
$\frac{1}{2} I$	$= 35° 15'$	log tan	$= 9.849254$
EA	$= 1012.52$	log	$= 3.005405$

1012.52×2	=	2025.04
Length 4° for 70° 30′	=	1762.50
Tangent route less original curve route	=	262.54

Then $262.54 : 266.06 :: 4° : 4.054°$ (= 4° 03.2′).

Hence each terminal curve will consist of 3° 24′ of 2° 01.6′ curve.

The new apex distance $A\,H$ will be

$$266.06 : 262.54 :: 1099.28 : 1084.74.$$

Similarly, the new $p = 2.487$.

In working out these proportions use logarithms.

For running in such a curve as a 4.054°, the decimal vernier (formerly supplied on transits by Young & Sons, of Philadelphia) is a great convenience. These instruments had one decimal vernier, the opposite one being of the usual sexagesimal form.

If, instead of the terminal angle 3° 24′, the final value of $p = 2.487$ be given, proceed as follows:

1st. Calculate the difference between the tangent and original curve routes; call this A.

2d. Multiply twice p by the tangent of half the whole intersection angle; call this product B. Then

$$A : A + B :: D_O : D_N$$

where D_O = degree of original main curve,
D_N = degree of new main curve,
$\frac{1}{2} D_N$ = degree of new one-chord.

Example. — Given a 4° curve for 70° 30′, to substitute a curve with spirals, retaining the same length of line and assuming $p = 2.487$.

From the preceding example, tangent route less the original curve route = 262.54 = A.

Also, $2p$	= 4.974	log =	0.696706
$\frac{1}{2} I$	= 35° 15′	log tan =	9.849254
B	= 3.52	log =	0.545960

$$A + B = 266.06$$

Then 262.54 : 266.06 :: 4° : 4.054° (= 4° 03.2′).

Other elements of the curves are found from formulas (1) and (10), as before.

Similarly, spirals may be inserted at the ends and between the members of a compound curve, while preserving the original length of line. This is shown by the following example, which also serves as a general review.

Given a compound curve, as follows (see Fig. 4):

Sta. 10 P. C. 4° R for 32° of angle = b.
Sta. 18 P. C. C. 10° R for 50° of angle = c.
Sta. 23 P. T.

Required to insert spirals at P. C., P. C. C., and P. T. without changing length of line.

Maximum speed on 10° curve = 31.6 miles per hour, which will also be taken on the 4°.

By Rule 2, length of one-chord spiral equals elevation in tenths of feet multiplied by speed. Hence, from Table III:

Length of one-chord for 10° curve = 31.6 × 5 = 158.0 ft.

Length of one-chord for 4° curve = 31.6 × 2 = 63.2 ft.

Length of one-chord between 4° and 10° = that for 10 − 4 = 6°.

Length of one-chord for 6° curve = 31.6 × 3 = 94.8 ft.

In this example radius = 5730 ÷ degree of curve.

The values of p are as follows: Since $p = R_M \times$ vers T_1,

Terminal ang. T_1 for 10° = 7° 54′, p = 5.44 = P.
Terminal ang. T_1 for 4° = 1° 16′, p = .34 = p.
Terminal ang. T_1 for 6° = 2° 51′, p = 1.18 = p_1

By formula (13):

$$WN = \frac{\left(\dfrac{5.44}{\cos 50^\circ} - 1.18\right)\cos 32^\circ - 0.34}{\sin 82^\circ} = 5.893.$$

By formula (14):

$$FW = (5.44 \times \tan 50^\circ) - 5.893 = .590.$$
$$TL = ML \times \sin b = EH \times \sin b.$$

$$EH = \frac{P}{\cos C} - p_1 = 7.283.$$

Hence 7.283 × sin 32° = 3.860 = TL.

Next calculate the effect of the shift GV measured along LT produced.

This will equal $WN \times \cos (b + c)$
$= 5.893 \times \cos 82° = 0.820 = TZ.$
To this add $TL = 3.860.$
Hence shift $LZ = 4.680.$

The notes of a new curve with one-chord spirals inserted, but retaining the original degrees of curve, would be as follows:

P. C. 4° = point *L* (Fig. 4) = Station		10 + 00.
Less *LZ*	=	04.68
Point *Z* (Fig. 4)	=	9 + 95.32
Deduct *GH* (Fig. 1) = 1432.5 × sin 1° 16′, [from (8)]	=	31.60
		9 + 63.72

Then

9 + 63.72	P. C.	2° one-chord for	1° 16′
+ 63.20			
10 + 26.92	P. C. C.	4° main curve for	28° 50′
7 + 20.83			
17 + 47.75	P. C. C.	7° one-chord for	6° 38′
94.80			
18 + 42.55	P. C. C.	10° main curve for	37° 22′
3 + 73.67			
22 + 16.22	P. C. C.	5° one-chord for	7° 54′
1 + 58			
23 + 74.22	P. T.		82° 00′

Thus far the procedure has been the same as though the change was to be made in the first line

prior to construction. It remains to reduce the above to a similar figure, having the same length between common points as the original line. For this purpose first calculate the original apex distances, A to L and F (Fig. 4), from the following formulas, A being the intersection of the main tangents through L and F produced (not shown in figure):

$$AL = R_2 \tan \tfrac{1}{2} I - \frac{(R_2 - R_1) \text{ vers } I_1}{\sin I}$$

$$AF = R_1 \tan \tfrac{1}{2} I + \frac{(R_2 - R_1) \text{ vers } I_2}{\sin I}$$

where R_2 is the larger radius $= 1432.5$
R_1 is the smaller radius $= 573.0$
I = the grand total angle $= 82°$
I_1 = the R_1 total angle c $= 50°$
I_2 = the R_2 total angle b $= 32°$

AL is on the side of the lighter curve and AF on that of the sharper.

Hence	AL =	935.21
	AF =	629.96
Original tangent route	=	1565.17
Original curve route (2300 – 1000)	=	1300.00
Difference	=	265.17

The new curve route =	23 + 74.22
less	9 + 63.72
or	14 + 10.50

The new tangent route is obtained thus: